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Self-Diffusion in a Two-Dimensional System of Colliding Vertical Sticks

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We consider a system of N horizontal lines where colliding vertical sticks are placed initially according to an equilibrium prescription; they move parallel to the lines and collide; the collisions take place between sticks of the same line and of the adjacent ones. The asymptotic behavior of a tagged stick is diffusive, and the self-diffusion constant is inversely proportional to N.

KEY WORDS: Infinite classical system; self-diffusion; invariance principle.

1. INTRODUCTION

The asymptotic motion of a test particle in a classical fluid is an interesting problem in nonequilibrium statistical mechanics. It is related to transport properties of matter, and the analysis of Brownian movement by Einstein[4] was perhaps the first deep insight into this subject. Rigorous analysis in this field has come relatively recently; the papers by Harris[(2)] and Spitzer[(3)] opened a new line of research, showing the asymptotically diffusive behavior of a test particle in a one-dimensional "sea" of identical particles at equilibrium.

It is to be noted that in this setting subtle phenomena may happen: by changing the initial state, Szász and Major[(4)] proved convergence to a Gaussian non-Wiener process, in spite of the convergence of the state to equilibrium. The situation for the multidimensional case is more complicated even at equilibrium: computer simulations give superdiffusive

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[4] See ref. 1, pp. 94–100 for a detailed account of Einstein's contribution to the study of Brownian motion.

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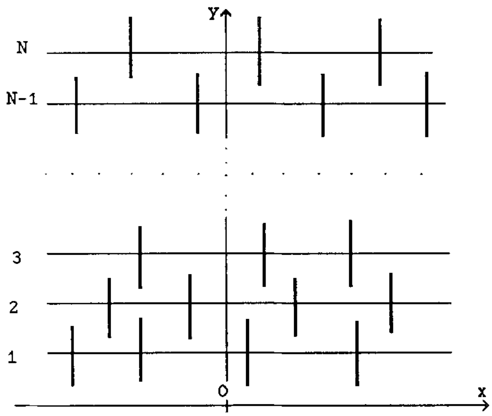

Fig. 1. A portion of a "generic" configuration of the system.

behavior for the two-dimensional hard-disk system, and diffusive behavior for the three-dimensional hard-sphere system. Here we consider the following model: on the plane $\mathbb{R}^2$ a fixed array of N parallel lines with constant unit spacing between them is given; the particle system consists of infinite identical sticks, orthogonal to these lines, on which their centers are free to move.

Their lengths are such that collisions between sticks are possible if they belong to the same or to adjacent lines. When two sticks collide, they exchange their velocities; otherwise they move freely (see Fig. 1). As usual, simultaneous collisions of more than two sticks will be ruled out by a suitable choice of the class of initial states. The following analysis shows a diffusive behavior for a test particle, with diffusion constant decreasing with N as $1/N$. The main point to be noted is the dependence of this quantity on the number N: in spite of the locality of the interaction, the test particle feels asymptotically a global (size) property of the system. A heuristic argument for this behavior is the following: the coupling between the lines is such that during its motion the test particle "drags" a whole set of $\simeq N$ sticks, placed on the other lines. In the next section we give more details on the model and state the results on the dynamics of the system and of the test particle; proofs and some auxiliary results are given in the third section.

2. ANALYSIS OF THE MODEL AND STATEMENT OF THE RESULTS

The initial states which we consider are as follows: positions on the lines are given according to independent Poisson point processes, with the same intensity (density) ρ (hereafter called ρ-Ppp); velocities are i.i.d. random variables for any stick with distribution function $dF(v)$ such that

$$\Pr\{v=0\} \equiv dF(\{0\})=0; \quad \mathbb{E}|v| \equiv \int |v|\, dF(v) < \infty; \quad \mathbb{E}v \equiv \int v\, dF(v)=0$$

A useful way to describe this model in a one-dimensional setting is the following: give on the line $\mathbb{R}^1$ positions $\{x_i\}_{i\in\mathbb{Z}}$, where the label i refers to the ordering

$$\dots x_{-1} < 0 \leqslant x_0 < x_1 < x_2 < \dots \tag{2.1}$$

according to an $N\rho$-Ppp. Then give velocities to each point as i.i.d. random variables with the common distribution $F(\alpha) \equiv \Pr(v \leqslant \alpha)$; call $\omega = \{(x_i, v_i)\}_{i\in\mathbb{Z}}$ the resulting phase configuration. Finally, give i.i.d. discrete "marks" $\sigma_i \in \{1, 2,\dots, N\}$, with uniform distribution: $\Pr(\sigma_i = \lambda) = 1/N$, $i\in\mathbb{Z}$, $\lambda \in \{1, 2,\dots, N\}$. Call $\tilde{\omega} = \{(x_i, v_i, \sigma_i))_{i\in\mathbb{Z}}$ the "marked" phase configuration point. The set of all ω (respectively $\tilde{\omega}$) will be called Ω (respectively $\tilde{\Omega}$). The phase configurations of our system correspond to the elements of $\tilde{\Omega}$ where the mark σ_i indicates the line, neglecting the zero-probability set, where coincidence of abscissas is present. Similarly to the hard-point system, where the collisional dynamics can be seen as "free motion plus exchange of labels at crossings," we can describe the evolution of the stick system in this way: consider the free evolution of points on $\mathbb{R}^1$,

$$x_k(t) = x_k + v_k t, \qquad v_k(t) = v_k, \qquad k\in\mathbb{Z} \tag{2.2}$$

Any point carries its label and mark until its trajectory crosses another one; then if the difference of marks is bigger than one, they go on freely without changes, otherwise they change marks and labels (see Fig. 2). That

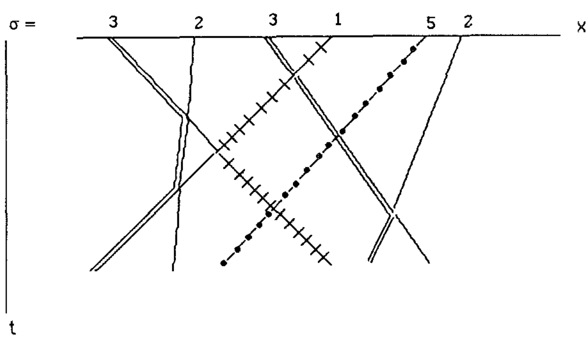

Fig. 2. Some diagrams $x = x(t)$ belonging to marked particles (3 and 2 collide, 3 and 1 freely cross).

is, if $\bar{t}>0$ is the first crossing time for particles labeled by $k, j, k \neq j$, i.e., $x_k(\mathbf{t}) = x_j(\bar{t})$ and no previous coincidences with other particles are verified for those particles between 0 and $\bar{t}$, then $\sigma_k(t) = \sigma_k$, $\sigma_j(t) = \sigma_j$ for $t \in [0, \bar{t})$; if $|\sigma_k - \sigma_j| > 1$, $\sigma_k(\bar{t}^+) = \sigma_k$, $\sigma_j(\bar{t}^+) = \sigma_j$; if $|\sigma_k - \sigma_j| \leqslant 1$, the particle with position and velocity $(x_k(\bar{t}^+), v_k(\bar{t}^+))$ has mark σ_j and vice versa. The similarity with the simpler hard-point system is now clear; in fact, for describing the collisional paths of the hard points it is sufficient to exchange labels in correspondence of crossings between free trajectories; it turns out that properties which do not depend on labels are studied by looking at the free system.

The same information now will be extracted from our system: we consider the free motion of particles plus exchange of marks according to the former rule, and we shall state a lemma on the invariance of the Poisson structure.

Lemma 1. States which are product of N independent ρ-Ppp are invariant under the time evolution.

While the proof is postponed to the next section, it will be useful to write down some formulas explaining the meaning of this statement. Namely, let us define: $\forall [a, b] \subset \mathbb{R}^1$, $\#_\lambda^{[a,b]}(t) \equiv$ number of particles with mark λ, $\lambda \in \{1, 2,..., N\}$, in $[a, b]$, at time $t \geqslant 0$; $\#_{\text{tot}}^{[a,b]}(t) \equiv \sum_{\lambda=1}^N \#_\lambda^{[a,b]}(t)$. The following relation, valid by hypothesis at $t=0$, holds for any positive time:

$$\Pr\{\#_1^{[a,b]}(t) = k_1, \#_2^{[a,b]}(t) = k_2,..., \quad \#_N^{[a,b]}(t) = k_N\}$$
$$= \prod_{i=1}^N \exp(-\rho(b-a)) \frac{[\rho(b-a)]^{k_i}}{k_i!} \tag{2.3}$$

This lemma implies this relevant property: the distances between sticks of the same line are independent exponentially distributed random variables, with expectation ρ^{-1}. We take as tagged particle the one in the first line, with the least nonnegative abscissa: $\forall \tilde{\omega} \in \tilde{\Omega}$, $\exists i_0 \equiv \min\{k\colon x_k \geqslant 0, \sigma_k = 1\}$; it will be the initial label of the test particle. The actual path of this particle will be called $Y(t) = y(t, \tilde{\omega})$: it depends (deterministically) on time and on the initial full configuration $\tilde{\omega}$, but as the ordering is not globally preserved, it will be somewhat more difficult to analyze it in terms of the associated free system. Nevertheless, considering $y(\cdot) = y(\cdot, \tilde{\omega}) \in C([0, T])$ (space of continuous real functions defined on $[0, T]$) as a stochastic process on $\tilde{\Omega}$, equipped with the full (i.e., with velocity distribution, too) product of the Ppps as probability measure, we formulate here the main result.

Theorem. The asymptotic scaled motion of a test particle in the stick system is diffusive, with diffusion constant $D_N = \mathbb{E}|v|/(N\rho)$; more precisely, $\forall \Phi(\cdot)$, continuous functional on $C([0, T])$,

$$\lim_{A \to \infty} \mathbb{E}\Phi(A^{-1/2} Y(A \cdot)) = \mathbb{E}\Phi(B(\cdot, B_N)) \tag{2.4}$$

where $B(\cdot, D_N)$ is the Brownian motion starting from zero with diffusion constant D_N.

3. PROOFS

Before going on to the proof of the lemma, we need to state some auxiliary propositions, related to the free dynamics of unmarked points, and then to the "free plus change of marks" dynamics. In the first proposition we consider a point system evolving freely on the line; its phase configurations can be considered realizations of a nonhomogeneous Ppp in the phase space $\mathbb{R}^2$ with intensity measure $\text{meas}(dv\, dx) = dF(v) \rho\, dx$. The free time evolution is very simple [see (2.2)]:

$$\omega(t) = \{(x_i + v_i t, v_i)\}_{i \in Z}, \qquad t \geqslant 0 \tag{3.1}$$

and leaves invariant, by inspection, the initial state. We want now to show a *cluster* property for this system.

Definition. For $T > 0$, $z \in \mathbb{R}$ is T-separating for ω if $\forall x_i \in \omega$, $\forall x_j \in \omega$, $x_i < z < x_j$, $\Rightarrow$ $x_i + v_i t < z < x_j + v_j t$, $\forall t \in [0, T]$.

Proposition 3.1. For all $[a, b] \subset \mathbb{R}$, for all $T > 0$ there exists $s_- \in \mathbb{Z}$, $s_+ \in \mathbb{Z}$ which are T-separating for almost all $\omega \in \Omega$, and such that $[s_-, s_+] \supset [a, b]$.

Proof. First of all, for any $z \in \mathbb{R}$ we can easily calculate the probability that it is T-separating. Let us define the following sets:

$$\begin{aligned} R_{z,T} \equiv {} & \{(x, v) : z - vT < x < z,\ v > 0\} \\ & \cup \{(x, v) : z < x < z - vT,\ v < 0\} \subset \mathbb{R}^2 \end{aligned} \tag{3.2}$$

$$\hat{R}_{z,T} = \{\omega : z \text{ is } T\text{-separating for } \omega\} \subset \Omega \tag{3.3}$$

then $\hat{R}_{z,T} = \{\omega$: number of points in $R_{z,T} \equiv \#^{R_{z,T}}(\omega) = 0\}$.

So

$$\begin{aligned} \Pr\{z \text{ is } T\text{-separating}\} &= \Pr\{\#^{R_{z,T}} = 0\} = \exp[-\text{meas}(R_{z,T})] \\ &= \exp(-T\rho\,\mathbb{E}|v|) \end{aligned} \tag{3.4}$$

Now we show that there are infinitely many T-separating integers.

Let S be the unit space translation acting on Ω:

$$S\{(x_i, v_i)\}_{i\in Z} = \{(x_i+1, v_i)\}_{i\in Z} \tag{3.5}$$

The Ppp on $\mathbb{R}^2$ is clearly mixing wrt S: $\forall \hat{A}, \hat{B}$, measurable $\subset \Omega$,

$$\Pr\{S^n\hat{A} \cap \hat{B}\} \xrightarrow[n\to\infty]{} \Pr\{\hat{A}\} \Pr\{\hat{B}\}$$

This implies an ergodic property[6] for the indicatrix function of the set $\hat{R}_{[b]+1,T}$:

$$\text{a.a. } \omega \in \Omega, \quad \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{\hat{R}_{[b]+1,T}}(S^{-k}\omega) \xrightarrow[n\to\infty]{} \Pr\{\hat{R}_{[b]+1,T}\} = \exp(-T\rho\,\mathbb{E}|v|)$$

[by (3.4)], but for the lhs we have

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{\hat{R}_{[b]+1,T}}(S^{-k}\omega) = \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{\hat{R}_{[b]+k+1,T}}(\omega)$$

The divergence of the sum $\sum_{k=0}^{n-1} \mathbf{1}_{\hat{R}_{[b]+k+1,T}}(\omega)$ for a.a. ω says that for infinitely many $k \in \mathbb{N}^+$, $\omega \in \hat{R}_{[b]+k+1,T}$.

The same argument works in the half-line $(-\infty, a)$. ▮

To prove Lemma 3.1, we shall use the following result about the invariance of the mark distribution during the evolution of a finite set of particles (the rather easy proof is omitted).

Let $\omega^M = \{(u_i, \omega_i)\}_{i\in 1,\ldots,M}$ be a phase configuration of M particles such that $u_1 < u_2 < \cdots < u_M$ and no multiple or simultaneous crossings are allowed in $[0, T]$.

Then give marks to each particle, independently and uniformly: let $\underline{\sigma}^M \equiv (\sigma_1, \sigma_2,\ldots, \sigma_M) \in \{1, 2,\ldots, N\}^M$, $\forall \underline{\lambda}^M \in \{1, 2,\ldots, N\}^M$, and $\Pr\{\underline{\sigma}^M = \underline{\lambda} = \underline{\lambda}^M \mid \omega^M\} = (1/N)^M$. The marks will change during the evolution as explained in Section 2. The invariance of the distribution means that the following holds:

$$\Pr\{\underline{\sigma}^M(t) = \underline{\lambda}^M \mid \omega^M\} = \left(\frac{1}{N}\right)^M, \qquad t \in [0, T] \tag{3.6}$$

Proof of Lemma 1. As the unmarked free systems preserves the $N\rho$-Ppp state, it will be sufficient to show that the distribution of marks for the particles which at time $t \in [0, T)$ are in a fixed interval $I \equiv [a, b]$ is the uniform-independent distribution given at time 0.

In fact, in this case, for $s \leqslant N$, $(i_1,..., i_s) \in \{1,..., N\}^s$, $k_i \in \mathbb{Z}^+$,

$$\Pr\{\#^I_{i_1}(t) = k_1,..., \#^I_{i_s}(t) = k_s \mid \#^I_{\text{tot}}(t) = L\} = \left(\frac{1}{N}\right)^{k_1 + \cdots + k_s} \frac{L!}{k_1! \cdots k_s!}$$

so that

$$\begin{aligned} &\Pr\{\#^I_1(t) = k_1,..., \#^I_N(t) = k_N\} \\ &\quad = \exp[-N\rho(b-a)] \frac{[N\rho(b-a)]^L}{L!} \left(\frac{1}{N}\right)^{k_1 + \cdots + k_N} \frac{L!}{k_1! \cdots k_N!} \\ &\quad = \prod_{i=1}^{N} \exp[-\rho(b-a)] \frac{[\rho(b-a)]^{k_i}}{k_i!} \end{aligned}$$

as $L = k_1 + \cdots + k_N$.

Let $x_{i_1}(t),..., x_{i_l}(t)$ be the positions of l particles ($l \geqslant 1$) in I at time t. Their marks $\sigma_{i_1}(t),..., \sigma_{i_l}(t)$ depend on the various crossings between their own trajectories and those of other particles coming from "everywhere" (i.e., even from outside I). So we are led to consider the finite set of particles staying in the least interval with T-separating extrema which contains $I \equiv [a, b]$.

For a given initial configuration ω and $T > 0$, let

$$s_- \equiv \sup\{s: s \leqslant a; s \text{ is } T\text{-separating}\} = s_-(\omega, I, T)$$

$$s_+ \equiv \inf\{s: s \geqslant b; s \text{ is } T\text{-separating}\} = s_+(\omega, I, T)$$

In fact, $[s_-, s_+] \supset I$ and

$$(s_+ - s_-) < \infty, \quad M(\omega) \equiv \#^{[s_-, s_+]}_{\text{tot}}(t) = \#^{[s_-, s_+]}_{\text{tot}} < \infty \quad \text{(a.s.)}$$

For the a.s. finite set of particles in $[s_-, s_+]$, we apply the preceding proposition: for any subset of such set (the ones belonging to I at time t) the distribution of marks is invariantly independent-uniform and (2.3) holds. ▌

Remark. The invariance of the ρ-Ppp state for a fixed mark (i.e., fixed line for the stick system) implies, as in the hard-point model, that during the actual evolution ther distances between successive sticks are stationary random variables (exponentially distributed with expectation ρ^{-1}).

We need now to give a further definition which will be useful in the sequel.

Definition. $\forall k \in \mathbb{Z}$, we define the *block in* $(k, k+1)$, with *center* $x_{i_k} \in (k, k+1)$, the following set $\hat{B}_k \subset \tilde{\Omega}$:

$$\begin{aligned}\hat{B}_k \equiv \{\tilde{\omega} \mid \#_{\text{tot}}^{(k,k+1)} = 2N-1;\\ k < x_{i_k - N+1} < \cdots < x_{i_k} < \cdots < x_{i_k + N-1} < k+1;\\ \sigma_{i_k \pm j}(\tilde{\omega}) = j+1, j = 0, 1,..., N-1\}\end{aligned} \tag{3.7}$$

Looking to the stick model, $\hat{B}_k$ is given by all phase configurations such that one stick in the first line is between two subsequent sticks of the second line; and these are subsequently nested in the same way up to the Nth line: all their abscissas are inside $(k, k+1)$ (see Fig. 3).

The following proposition says that in a typical configuration there are infinitely many such structures.

Proposition 3.3. Denoting by Pr the probability measure on $\tilde{\Omega}$ induced by the product Ppp, the following holds:

$$\Pr\{\hat{B}_k \text{ i.o.}\} = 1 \tag{3.8}$$

Proof. First of all let us compute for a given $k \in \mathbb{Z}$, the probability of occurrence of $\hat{B}_k$.

The Poisson structure of the state allows an explicit evaluation:

$$\begin{aligned}\Pr\{\hat{B}_k\} &= \left(\frac{1}{N}\right)^{2N-1} \exp(-N\rho) \frac{(N\rho)^{2N-1}}{(2N-1)!}\\ &= \exp(-N\rho) \frac{(\rho)^{2N-1}}{(2N-1)!} = p_N > 0\end{aligned}$$

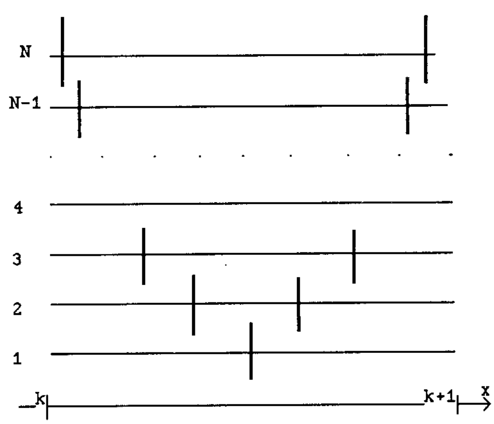

Fig. 3. A configuration belonging to the block $\hat{B}_k$ (only sticks with abscissas in $[k, k+1]$ are depicted).

Moreover, for $k \neq k'$, $\hat{B}_k$ and $\hat{B}_{k'}$ are independent; then, via the Borel–Cantelli lemma for independent events, (3.8) holds. ∎

Remarks. (i) Let us define $k^+(\tilde{\omega})$, the index of the *first positive block*:

$$k^+ = k^+(\tilde{\omega}) = \inf\{k \in \mathbb{Z}^+ \mid \tilde{\omega} \in \hat{B}_k\} \tag{3.9}$$

From independence we have

$$\Pr\{k^+ = j\} = (1 - p_N)^{j-1} p_N, \qquad j = 1, 2,...$$

Consequently,

$$\mathbb{E}(k^+) = 1/p_N \tag{3.10}$$

(ii) Analogous definition [as in (3.9)] and result hold for the *first negative block*, with index $k^- \equiv \sup\{k \in \mathbb{Z}^-: \tilde{\omega} \in \hat{B}_k\}$, and *center* initially in $x_{i_{k^-}}$. These two blocks (resp. centers) will be called the *first* ones.

(iii) We observe that these structures will be not completely preserved during the collisional evolution, but the ordering among the $2N-1$ particles which initially define the blocks is preserved.

The next proposition is related to the behavior of the centers under the real (collisional) dynamics; $\forall j \in \mathbb{Z}$, the continuous function $t \to y_j(t)$, with $y_j(0) = x_j(\tilde{\omega})$, $\dot{y}_j(0) = v_j(\tilde{\omega})$, represents the actual path of the particle initially labeled by j.

Proposition 3.4. The joint scaled motions of the first centers converge (weakly as processes in $C[0, T]$) to the same Brownian motion with diffusion constant $D_N = \mathbb{E}|v|/N\rho$.

Proof. Let us consider the one-dimensional hard-point system associated to ours; if $\tilde{\omega} = \{(x_k, v_k, \sigma_k)\}_{k \in \mathbb{Z}}$ is the initial (marked) phase configuration, its evolution is defined to be purely collisional, independently of the marks; i.e., the associated system is the hard-point system starting from ω.

For any $k \in \mathbb{Z}$, let $Y_k(t) = Y_k(t \mid \tilde{\omega}) = Y_k(t \mid \omega)$ be the hard-point collision path starting in $x_k(\tilde{\omega})$, with velocity $v_k(\tilde{\omega})$; $Y_0(t)$ has a known asymptotic behavior[3,5]: $A^{-1/2} Y_0(A\,\cdot)$ converges (weakly, as a process in $C[0, T]$), to $B(\cdot, D_N)$; the key observation is the following: for almost all initial $\tilde{\omega}$,

$$y_{i_{k\pm}}(t) = Y_{i_{k\pm}}(t), \qquad t \geqslant 0$$

In fact the path of a center of any block is never crossed by any other path, and it can be traced in the set of free trajectories exactly as in the hard-point model: by construction any stick on its left (right) will remain forever on its left (right). Now we can adapt the argument used by Szász[7]: in that case the author considered two points starting from *deterministic*, *A-dependent positions*; the limit processes, if the initial spacing is infinitesimal w.r.t. the divergent parameter A, coincide with probability one. Moreover, the choice of the initial state gives that this limit process is $B(\cdot, D_N)$.

In fact, let $Z^A_\pm(\cdot)$ be scaled paths in the hard-point system starting from initial positions diverging to $\pm\infty$ as A^α, $\alpha \in (0, 1)$; we have

$$Z^A_-(t) \leqslant Y^A_0(t) \leqslant Z^A_+(t) \tag{3.11}$$

and we know from ref. 7 that $Z^A_-(\cdot)$, $Y^A_0(\cdot)$, and $Z^A_+(\cdot)$ are converging to the *same* process $(B(\cdot, D_N))$. The random initial positions of the two first centers are finite a.s.: under scaling they go to zero with probability one because [see (3.10)]:

$$\mathbb{E}\,|x_{i_{k\pm}}| \leqslant \mathbb{E}(|k^\pm| + 1) \leqslant 1 + \frac{1}{p_N} \quad \blacksquare$$

Proof of the Theorem. We shall compare the path of the tagged stick with the paths of the nearest centers. Let the initial position of the tagged stick be x_{i_0}; we shall introduce a shorter notation for our scaled processes:

$$A^{-1/2} y_{i_{k\pm}}(At) \equiv y^A_\pm(t); \qquad A^{-1/2} y(At) \equiv y^A(t)$$

Then for the path $y(t) \equiv y_{i_0}(t)$, which describes the motion of the tagged stick in the actual *stick* dynamics, the following is true:

$$Y_{i_{k-}}(t) = y_{i_{k-}}(t) \leqslant y(t) \leqslant y_{i_{k+}}(t) = Y_{i_{k+}}(t) \tag{3.12}$$

This double inequality goes on the scaled processes:

$$y^A_-(t) \leqslant y^A(t) \leqslant y^A_+(t) \tag{3.13}$$

As the side processes in (3.13) converge to the same process (Proposition 3.4), the same is true for $y^A(t)$; this is due to the fact that a.s.:

$$\sup_{t \in [0,T]} (y^A(t) - y^A_-(t)) \leqslant \sup_{t \in [0,T]} (y^A_+(t) - y^A_-(t)) \quad \blacksquare$$

4. CONCLUSIONS

The result is mainly based on the following point: on the first (any) line there are sticks which move as though they were hard points in a gas of density $N\rho$. This fact is essentially due to the presence of ordered structures (blocks) which act like *heavy walls* during the motion of the sticks and are randomly but a.s. finitely spaced. Then the structure of he process forces any other stick in that line to move asymptotically in the same way.

The infinite ($N = \infty$) case needs different techniques: the associated one-dimensional system obtained by projection is ill-defined; on the other hand, the particle cannot move more freely than in a finite array of N lines ($\forall N$): so one is led to conjecture a subdiffusive behavior and to look for a suitable space-time scaling leading to a nontrivial limit.

Finally, we wish to list some related models, somewhat nearer to physics, where other ideas are needed, too:

(i) A system of vertical hard sticks moving in both directions in the plane.

(ii) The same system as in (i), but enclosed in a vertical slab with stochastic boundary conditions. In this case the main interest is the convergence of the state to the stationary one driven by the boundary conditions: at the kinetic level (i.e., in the Boltzmann–Grad limit) some results are known.[8]

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